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1984 J. Phys. A: Math. Gen. 17 2351

(http://iopscience.iop.org/0305-4470/17/11/030)

J. Phys. A: Math. Gen. 17 (1984) 2351–2353. Printed in Great Britain

COMMENT

The Ising spin-glass near the de Almeida–Thouless line

Hans-Jürgen Sommers

Institut für Festkörperforschung, Kernforschungsanlage Jülich, D-5170 Jülich, Federal Republic of Germany

Received 23 March 1984

Abstract. We confirm that the spin-glass transition at the AT line is a third-order transition in contradiction to the result of Elderfield. We calculate the discontinuities of the first thermodynamic derivatives of the total susceptibility along the AT line. The local susceptibility has continuous derivatives except at $T = T_c$ for zero external field.

Elderfield (1983) has calculated properties of the long-ranged Ising spin-glass near the AT line (de Almeida and Thouless 1978) using an expansion derived previously by the author (Sommers 1983, referred to as I). Elderfield claims that the magnetisation has a weak cusp near the AT line. This would imply that the transition is of second order in the Ehrenfest scheme. One sees immediately that equations (25), (36) and (37) of that paper contain errors. On the other hand, in I it was claimed that the transition is of third order in agreement with Toulouse *et al* (1982). Unfortunately, there are misprints in equations (39) and (43) of I. These errors are corrected below and a simplified derivation is provided. In addition we calculate the discontinuities of third derivatives of the free energy along the AT line.

In I the free energy was expanded up to third order in $\Delta'(x)$ and $q'(x)$. It turns out that up to this order

$$\Delta(x) \propto q(1) - q(x). \tag{1}$$

If we use this relation the free energy remains a stationary function of three order parameters, $q(0)$, $q(1)$, $\Delta(0)$. Let us define $q = q(0)$, $\Delta = \Delta(0)$, $g = q(1) - q(0) - \Delta(0)$ and $\varepsilon = (\beta J)^2/2$. Then the free energy, f, in an external field, b, at temperature $1/\beta$ can be written as

$$\begin{aligned} -\beta f = \log 2 - \tfrac{1}{2}\overline{\log(1-m^2)} + \tfrac{1}{2}\varepsilon(1-q^2) + \varepsilon g(1-\overline{m^2}) + \tfrac{1}{2}\varepsilon[g^2(1+\varepsilon\overline{m^{(3)}}) + \Delta g(1-2\varepsilon\overline{m'^2})] \\ + \varepsilon^3[(g^3/3!)\overline{m^{(5)}} + \tfrac{8}{3}\Delta g^2\,\overline{m^2 m'^2} + 4\Delta g^2\overline{(1-5m^2)m'^2} + \tfrac{4}{3}\Delta^2 g\,\overline{(1-3m^2)m'^2}] \end{aligned} \tag{2}$$

with $m = \tanh(\beta b + \beta J z\sqrt{q})$ and $m' = 1 - m^2$. The bar means the average over the Gaussian variable z with $\bar{z} = 0$ and $\overline{z^2} = 1$. I would like to stress that this expansion is valid only up to lowest non-trivial order near the AT line excluding the point $T = T_c$ $(b = 0)$. For $b = 0$, $\Delta'(x)$ and $q'(x)$ are not of the same order of magnitude and therefore next-order terms have to be taken into account (Parisi 1979a). Condition (1) implies generally that Parisi's differential equation (De Dominicis *et al* 1982) can be solved and the result leads to the three-order-parameter theory of Parisi (1979b) which we have shown to be exact in the neighbourhood of the AT line.

Let us expand around the Sherrington–Kirkpatrick solution (1975), $q = q_0 + \delta$, where throughout the subscript 0 indicates the SK value:

$$-\beta f = -\beta f_0 + \tfrac{1}{2}\varepsilon[(\delta+g)^2\rho_0 + \Delta g\eta_0] + \varepsilon^3\{[(\delta+g)^3/3!]\overline{m_0^{(5)}} + \tfrac{8}{3}\Delta g^2\overline{m_0^2 m_0'^2}$$
$$+4\Delta g(\delta+g)\overline{(1-5m_0^2)m_0'^2} + \tfrac{4}{3}\Delta^2 g\overline{(1-3m_0^2)m_0'^2}\} \tag{3}$$

with

$$\eta_0 = 1-(\beta J)^2\overline{m_0'^2}, \tag{4}$$

$$\rho_0 = 1-(\beta J)^2\overline{(1-3m_0^2)m_0'}. \tag{5}$$

η_0 and ρ_0 are the eigenvalues occurring in the de Almeida–Thouless stability analysis (1978). The AT line is characterised by $\eta_0 = 0$. For the multicritical point, $T = T_c$, we have in addition $\rho_0 = 0$, which means $q_0 = 0$ or $b = 0$. Along the AT line we can neglect $\delta + g$ which is of order η_0^2. We find for the values at the stationary point

$$g = -\eta_0/16\varepsilon^2\overline{m_0^2 m_0'^2}, \tag{6}$$

$$\Delta = -\eta_0/8\varepsilon^2\overline{(1-3m_0^2)m_0'^2}, \tag{7}$$

$$-\beta f = -\beta f_0 + \varepsilon g\Delta\eta_0/6. \tag{8}$$

In order to compare with I, we express η_0 by $\eta = 1-2\varepsilon\overline{m'^2}$ appearing in (2). The relation to η_0 is given by

$$\eta = 1-2\varepsilon\overline{m'^2} = \eta_0 - 8\varepsilon^2 g\overline{(1-5m_0^2)m_0'^2}$$
$$= \eta_0\overline{(1-3m_0^2)m_0'^2}/(2\overline{m_0^2 m_0'^2}). \tag{9}$$

Equations (6)–(8) together with (9) replace (39), (42), (43) of I.

We see that the difference $f - f_0$ is of third order in η_0, the deviation from the AT line. Thus the transition is of third order. The difference of the magnetisation is of order η_0^2. Using the stationary property of (3), we are able to calculate the magnetisation up to order η_0^3. Instead, let us calculate the discontinuities of the thermodynamic derivatives of the total susceptibility along the AT line

$$(\partial/\partial b)(\chi-\chi_0) = (\varepsilon g\Delta/\beta\eta_0^2)(\partial\eta_0/\partial b)^3, \tag{10}$$

$$(\partial/\partial T)(\chi-\chi_0) = (\varepsilon g\Delta/\beta\eta_0^2)(\partial\eta_0/\partial b)^2\partial\eta_0/\partial T. \tag{11}$$

The discontinuities of the various other third derivatives of the free energy are simply related to these. In addition along the AT line we have

$$\mathrm{d}b/\mathrm{d}T = -(\partial\eta_0/\partial T)/(\partial\eta_0/\partial b). \tag{12}$$

For $T \to 0$, we find

$$(\partial/\partial b)(\chi-\chi_0) \simeq (25/32J^2)(Tb/J^2)^3, \tag{13}$$

$$(\partial/\partial T)(\chi-\chi_0) \simeq (25/32J^2)(Tb/J^2)^2, \tag{14}$$

with

$$T/J \simeq \tfrac{4}{3}(2\pi)^{-1/2}\exp[-\tfrac{1}{2}(b/J)^2] \tag{15}$$

and $\mathrm{d}T/\mathrm{d}b = -Tb/J^2$. Corrections are of relative order $(Tb/J^2)^2$ and $(T/J)^2$. The prefactor in (13), (14) is the discontinuity of the second temperature derivative of the entropy. For $T \to J = T_c$, we find

$$(\partial/\partial b)(\chi - \chi_0) \simeq \beta^2 \tfrac{4}{3}[\tfrac{1}{3}(1 - T/T_c)]^{1/2}, \tag{16}$$

$$(\partial/\partial T)(\chi - \chi_0) \simeq \beta^2 \tfrac{4}{3}(1 - T/T_c), \tag{17}$$

with

$$\mathrm{d}b/\mathrm{d}T \simeq -[3(1 - T/T_c)]^{1/2} \tag{18}$$

in agreement with Toulouse *et al* (1982). The discontinuity of the second temperature derivative of the entropy is given by $\beta^2 4(1 - T/T_c)^2$. It is interesting to compare with the SK values along the AT line for $T \to T_c$

$$(\partial/\partial b)\chi_0 \simeq -\beta^2\, 6[\tfrac{1}{3}(1 - T/T_c)]^{1/2}, \tag{19}$$

$$(\partial/\partial T)\chi_0 \simeq -\beta^2 \tfrac{4}{3}(1 - T/T_c). \tag{20}$$

Corrections are of relative order $(1 - T/T_c)$. Thus, as a function of temperature, χ_0 reaches its maximum slightly below the AT line. In the spin-glass phase, the slopes are reduced. Near T_c, χ is independent of temperature in agreement with the Parisi–Toulouse hypothesis (1980). It is important to note that the first derivatives of the local susceptibility are continuous except for $b = 0$. The transition at the AT line is related to the off-diagonal susceptibility, which is reflected in the divergence of the Edwards–Anderson susceptibility.

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